How to implement the belief functions

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Plan

- Natural order
- Smets codes
- General framework
- How to obtain bbas?
  - Random bbas
  - Distance based model
  - probabilistic based model

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Natural order or Binary order

Discernment frame: $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$
Power set: all the disjunctions of Θ:
$2^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, \{\theta_1 \cup \theta_2\}, \dots, \Theta\}$

Natural order:

$$
\begin{aligned}
2^\Theta = \; & \{\emptyset, \{\theta_1\}, \{\theta_2\}, \{\theta_1 \cup \theta_2\}, \\
& \{\theta_3\}, \{\theta_1 \cup \theta_3\}, \{\theta_2 \cup \theta_3\}, \{\theta_1 \cup \theta_2 \cup \theta_3\}, \\
& \{\theta_4\}, \dots, \Theta\}
\end{aligned}
$$

Natural order or Binary order

Natural order:

$\emptyset$ 0	θ_1 1	θ_2 2	$\theta_1 \cup \theta_2$ $3 = 2^2 - 1$
θ_3 $4 = 2^{3-1} + 1$	$\theta_1 \cup \theta_3$ 5	$\theta_2 \cup \theta_3$ 6	$\theta_1 \cup \theta_2 \cup \theta_3$ $7 = 2^3 - 1$
θ_4 $8 = 2^{4-1} + 1$	$\dots$	$\dots$	$\dots$
θ_i $2^{i-1} + 1$	$\dots$	$\dots$	Θ 2^n

Natural order or Binary order

Bba in Matlab:

Example: $m_1(\theta_1) = 0.5$, $m_1(\theta_3) = 0.4$, $m_1(\theta_1 \cup \theta_2 \cup \theta_3) = 0.1$
$m_2(\theta_3) = 0.4$, $m_2(\theta_1 \cup \theta_3) = 0.4$

```
F1=[1 4 7]';
F2=[4 5]';
M1=[0.5 0.4 0.1]';
M2=[0.4 0.6]';
```

Natural order or Binary order

Combination

$$m_{\text{Conj}}(X) = \sum_{Y_1 \cap Y_2 = X} m_1(Y_1) m_2(Y_2) \quad (1)$$

$\theta_1 \cap (\theta_1 \cup \theta_3)$: $1 \cap 5$
In binary on base 3: 1=100 and 5=101=100 |001
$100\&101 = 100$

Natural order or Binary order

In Matlab:

```
sizeDS=3;
F1=[1 4 7]';
F2=[4 5]';
M1=[0.5 0.4 0.1]';
M2=[0.4 0.6]';
Fres=[];
Mres=[];
for i=1:size(F1)
   for j=1:size(F2)
      Fres=[Fres bi2de(de2bi(F1(i),sizeDS)&de2bi(F2(j),sizeDS))];
      Mres=[Mres M1(i)*M2(j)];
   end
end
```

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Smets codes for DST

Smets gaves the codes of the Mobius transform (see Only_Mobius_Transf) for conversions:

- bba and belief: mtobel, beltom
- bba and plausibility: mtopl, pltom
- bba and communality: mtoq, qtom
- bba and implicability: mtob, btom
- bba to pignistic probability: mtobetp
- etc...

e.g. in Matlab:
m1=[0 0.4 0.1 0.2 0.2 0 0 0.1]';
mtobel(m1)
gives: 0 0.4000 0.1000 0.7000 0.2000 0.6000 0.3000 1.0000

Conjunctive combination

$$m_{\text{Conj}}(X) = \sum_{Y_1 \cap \ldots \cap Y_m = X} \prod_{j=1}^{m} m_j(Y_j)$$

The practical way:

$$q(X) = \prod_{j=1}^{m} q_j(X)$$

Disjunctive combination

$$m_{\text{Dis}}(X) = \sum_{Y_1 \cup \ldots \cup Y_m = X} \prod_{j=1}^{m} m_j(Y_j)$$

The practical way:

$$b(X) = \prod_{j=1}^{m} b_j(X)$$

In Matlab

For the conjunctive rule of combination:

```
m1=[0 0.4 0.1 0.2 0.2 0 0 0.1]';
m2=[0 0.2 0.3 0.1 0.1 0 0.2 0.1]';
q1=mtoq(m1);
q2=mtoq(m2);
qConj=q1.*q2;
mConj=qtom(qConj)
mConj =
  0.4100 0.2200 0.2000 0.0500 0.0900 0 0.0200 0.0100
```

In Matlab

For the disjunctive rule of combination:

```
m1=[0 0.4 0.1 0.2 0.2 0 0 0.1]';
m2=[0 0.2 0.3 0.1 0.1 0 0.2 0.1]';
b1=mtob(m1);
b2=mtob(m2);
bConj=b1.*b2;
bDis=b1.*b2;
mDis=btom(bDis)
mDis =
   0 0.0800 0.0300 0.3100 0.0200 0.0800 0.1300 0.3500
```

DST code

Once bbas are combined, to decide just use the functions mtobel, mtopl or mtobetp, etc.

In Matlab

mtopl(mConj)

0 0.2800 0.2800 0.5000 0.1200 0.3900 0.3700 0.5900

mtobetp(mConj)

0.4209 0.4040 0.1751

mtopl(mDis)

0 0.8200 0.8200 0.9800 0.5800 0.9700 0.9200 1.0000

mtobetp(mDis)

0.3917 0.3667 0.2417

DST code

DST code for the combination:

- criteria=1 Smets criteria
- criteria=2 Dempster-Shafer criteria (normalized)
- criteria=3 Yager criteria
- criteria=4 disjunctive combination criteria
- criteria=5 Dubois criteria (normalized and disjunctive combination)
- criteria=6 Dubois and Prade criteria (mixt combination)
- criteria=7 Florea criteria
- criteria=8 PCR6
- criteria=9 Cautious Denoeux Min for non-dogmatics functions
- criteria=10 Cautious Denoeux Max for separable functions
- criteria=11 Hard Denoeux for functions sous-normales
- criteria=12 Mean of the bbas

PCR6 (Martin et Osswald, 2006, 2007)

Transfers the partial conflict on focal elements given this conflict proportionnaly to the masses.

$$m_{\mathrm{PCR5}}(X) = m_{\mathrm{Conj}}(X) + \sum_{\substack{Y \in 2^\Theta,\\ X \cap Y = \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right)$$

$$\begin{aligned} m_{\mathrm{PCR6}}(X) &= m_{\mathrm{Conj}}(X) \\ &+ \sum_{j=1}^{s} m_i(X)^2 \sum_{\substack{\bigcap_{j'=1}^{s-1} Y_{\sigma_j(j')} \cap X = \emptyset \\ (Y_{\sigma_j(1)}, \ldots, Y_{\sigma_j(s-1)}) \in (2^\Theta)^{s-1}}} \left(\frac{\prod_{j'=1}^{s-1} m_{\sigma_j(j')}(Y_{\sigma_j(j')})}{m_j(X) + \sum_{j'=1}^{s-1} m_{\sigma_j(j')}(Y_{\sigma_j(j')})} \right) \end{aligned}$$

DST code

decisionDST code for the decision:

- criteria=1 maximum of the plausibility
- criteria=2 maximum of the credibility
- criteria=3 maximum of the credibility with rejection
- criteria=4 maximum of the pignistic probaiblity
- criteria=5 Appriou criteria

DST code

test.m:

```
m1=[0 0.4 0.1 0.2 0.2 0 0 0.1]';
m2=[0 0.2 0.3 0.1 0.1 0 0.2 0.1]';
m3=[0.1 0.2 0 0.4 0.1 0.1 0 0.1]';

m3d=discounting(m3,0.95);

M_comb_Smets=DST([m1 m2 m3d],1);
M_comb_PCR6=DST([m1 m2],8);

class_fusion=decisionDST(M_comb_Smets',1)
class_fusion=decisionDST(M_comb_PCR6',1)
class_fusion=decisionDST(M_comb_Smets',5,0.5)
```

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General framework

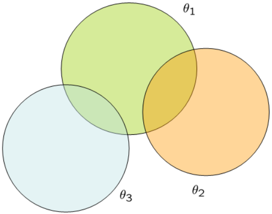
- Main problem of the DST code: all element must be coded (not only the focal elements)
- Only usable for belief functions defined on power set (2^Θ)
- General belief functions framework works for power set and hyper power set (D^Θ)

DSmT introduced by Dezert, 2002.

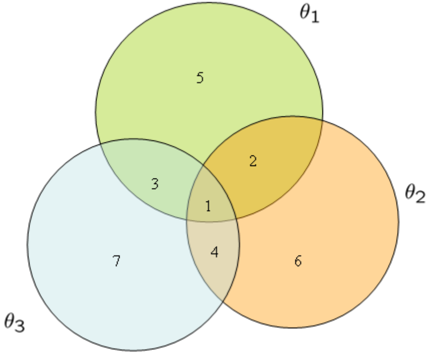
- D^Θ closed set by union and intersection operators
- D^Θ is not closed by complementary, $A \in D^\Theta \nRightarrow \overline{A} \in D^\Theta$
- if $|\Theta| = n$: $2^\Theta << D^\Theta << 2^{2^\Theta}$
- D_r^Θ: reduced set considering some constraints $(\theta_2 \cap \theta_3 \equiv \emptyset)$

$$\mathrm{GPT}(X) = \sum_{Y \in D_r^\Theta, Y \not\equiv \emptyset} \frac{\mathcal{C}_\mathcal{M}(X \cap Y)}{\mathcal{C}_\mathcal{M}(Y)} m(Y)$$

where $\mathcal{C}_\mathcal{M}(X)$ is the cardinality of X in D_r^Θ



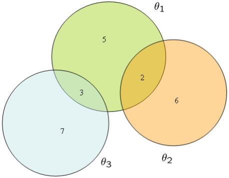
A simple codification (Martin, 2009)
Affect an integer of $[1; 2^n - 1]$ to each distinct part of Venn diagram de Venn ($n = |\Theta|$)



$$\Theta = \{[1\ 2\ 3\ 5], [1\ 2\ 4\ 6], [1\ 3\ 4\ 7]\}$$

Adding a constraint: if $\Theta = \{[1\ 2\ 3\ 5], [1\ 2\ 4\ 6], [1\ 3\ 4\ 7]\}$ and we know $\theta_2 \cap \theta_3 \equiv \emptyset$ (*i.e.* $\theta_2 \cap \theta_3 \notin D_r^\Theta$)
The parts 1 and 4 of Venn diagram does not exist:

$$\Theta_r = \{[2\ 3\ 5], [2\ 6], [3\ 7]\}$$



Operations on focal elements

$$\theta_1 \cap \theta_3 = [3]$$
$$\theta_1 \cup \theta_3 = [2\ 3\ 5\ 7]$$
$$(\theta_1 \cap \theta_3) \cup \theta_2 = [2\ 3\ 6]$$

The cardinality $\mathcal{C}_\mathcal{M}(X)$: the number of intergers in the codification of X

Gives an easy Matlab programmation of the combination rules and the decision functions

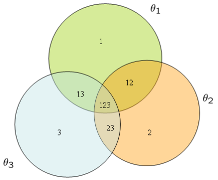
General framework: codification

Decoding: to present the decision or a result to the human - *The codification is not understandable*

If the decision set is given, we just have to sweep the corresponding part of D_r^Θ

Without any knowledge of the element to decode:

1. We can use the Smarandache condification more lisible but less practical in Matlab



2. We sweepp all D_r^Θ (first considering 2^Θ). There is a combinatorial risk.

Description of the problem

```
CardTheta=3; % cardinality of Theta
% list of experts with focal elements and associated bba
   expert(1).focal='1' '1u3' '3' '1u2u3';
   expert(1).bba=[0.5 0.3 0.1 0.1];

   expert(2).focal='1' '2' '1u3' '1u2u3';
   expert(2).bba=[0.5 0.6 0.1 0.1];

   expert(3).focal='1' '3n2' '(1n2)u3';
   expert(3).bba=[0.2 0.7 0.1];

constraint=''; % set of empty elements e.g. '1n2'
```

In test.m

Description of the problem

elemDec='A'; % set of decision elements:

- list of elements on which we can decide,
- A for all,
- S for singletons only,
- F for focal elements only,
- SF for singleton plus focal elements,
- Cm for given specificity, e.g. elemDec='Cm' '1' '4'; minimum of cardinality 1, maximum=4,
- 2T for only 2^Θ (DST case)

Parameters

Combination citerium

criteriumComb = is the combination criterium

- criteriumComb=1 Smets criterium
- criteriumComb=2 Dempster-Shafer criterium (normalized)
- criteriumComb=3 Yager criterium
- criteriumComb=4 disjunctive combination criterium
- criteriumComb=5 Florea criterium
- criteriumComb=6 PCR6
- criteriumComb=7 Mean of the bbas

Parameters

Combination citerium

criteriumComb = is the combination criterium

- criteriumComb=8 Dubois criterium (normalized and disjunctive combination)
- criteriumComb=9 Dubois and Prade criterium (mixt combination)
- criteriumComb=10 Mixt Combination (Martin and Osswald criterium)
- criteriumComb=11 DPCR (Martin and Osswald criterium)
- criteriumComb=12 MDPCR (Martin and Osswald criterium)
- criteriumComb=13 Zhang's rule

Parameters

Decision citerium

criteriumDec = is the combination criterium

- criteriumDec=0 maximum of the bba
- criteriumDec=1 maximum of the pignistic probability
- criteriumDec=2 maximum of the credibility
- criteriumDec=3 maximum of the credibility with reject
- criteriumDec=4 maximum of the plausibility
- criteriumDec=5 Appriou criterium
- criteriumDec=6 DSmP criterium

Parameters

Mode of fusion

mode='static'; % or 'dynamic'

Display

display = kind of display

- display = 0 for no display,
- display = 1 for combination display,
- display = 2 for decision display,
- display = 3 for both displays

Fusion

fuse(expert,constraint,CardTheta,criteriumComb,criteriumDec,mode, elemDec,display)

Called functions:

- Coding: call coding, addConstraint, codingExpert
- Combination: call combination
- Decision: call decision
- Display: call decodingExpert, decodingFocal

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In Matlab:

1. ThetaSize=3;
2. nbFocalElement=4;
3. ind=randperm(2^ThetaSize);
4. indFocalElement=ind(1:nbFocalElement);
5. randMass=diff([0; sort(rand(nbFocalElement-1,1)); 1]);
   We take the difference between 3 ordered random number in [0,1], e.g. diff([0; [0.3; 0.9] ; 1]) gives 0.3 0.6 0.1
6. MasseOut(indFocalElement,i)=randMass;

Random bbas

- focal elements can be evrywhere:
  ind=randperm(2^ThetaSize);
- focal elements not on the emptyset:
  ind =1+randperm(2^ThetaSize-1);
- no dogmatic mass: one focal element is on Theta (ignorance):
  ind =[2^ThetaSize randperm(2^ThetaSize-1)];
- no dogmatic mass: one focal element is on Theta (ignorance) and focal elements are not on the emptyset
  ind =[2^ThetaSize (1+randperm(2^ThetaSize-2))]
  (nbFocalElement==ThetaSize):
- all the focal elements are the singletons:

```
ind=[ ];
for i=1:ThetaSize
   ind=[ind; 1+2^(i-1)];
end
```

Distance based model (Denœux 1995)

Only θ_i and Θ are focal elements, $n * m$ sources (experts)

- Prototypes case ($\mathbf{x}_i$ center of θ_i). For the observation x

$$\begin{aligned} m_j^i(\theta_i) &= \alpha_{ij} \exp[-\gamma_{ij} d^2(x, \mathbf{x}_i)] \\ m_j^i(\Theta) &= 1 - \alpha_{ij} \exp[-\gamma_{ij} d^2(x, \mathbf{x}_i)] \end{aligned}$$

  - $0 \leq \alpha_{ij} \leq 1$: discounting coefficient and $\gamma_{ij} > 0$, are parameters to play on the quantity of ignorance and on the form of the mass functions
  - The distance allows to give a mass to x higer according to the proximity to θ_i
- belief k-nn: we consider the k-nearest neigborhs insteed to $\mathbf{x}_i$
- Then we combine the bbas

In Matlab (Denœux codes)

See ExampleIris.m

```
load iris
ind=randperm(150);
xapp=x(ind(1:100),:);
Sapp=S(ind(1:100));
xtst=x(ind(101:150),:);
Stst=S(ind(101:150));
[gamm,alpha] = knndsinit(xapp,Sapp); % initialization
[gamm,alpha,err] = knndsfit(xapp,Sapp,5,gamm,0); % parameter
                                                      optimization
[m,L] =knndsval(xapp,Sapp,5,gamm,alpha,0,xtst); % test
[value,Sfind]=max(m);
[mat_conf,vect_prob_classif,vect_prob_error]=build_conf_matrix(Sfind,Stst)
```

Probabilistic based model $^{(1/6)}$

- Need to estimate $p(S_j|\theta_i)$
- 2 models proposed by Appriou according to both axioms:
  1. the $n * m$ couples $[M_i^j, \alpha_{ij}]$ are distinct information sources where focal elements are: θ_i, θ_i^c and Θ
  2. If $M_i^j = 0$ and the information is valid ($\alpha_{ij} = 1$) then it is certain that θ_i is not true.

Model 1: $m_j^i(\theta_i) = M_i^j$
$m_j^i(\theta_i^c) = 1 - M_i^j$

Model 2: $m_j^i(\Theta) = M_i^j$
$m_j^i(\theta_i^c) = 1 - M_i^j$

Adding the reliability α_{ij} with the discounting:
Model 1:

$$\begin{array}{rcl} m_j^i(\theta_i) & = & \alpha_{ij} M_i^j \\ m_j^i(\theta_i^c) & = & \alpha_{ij}(1 - M_i^j) \\ m_j^i(\Theta) & = & 1 - \alpha_{ij} \end{array}$$

Modele 2:

$$\begin{array}{rcl} m_j^i(\theta_i) & = & 0 \\ m_j^i(\theta_i^c) & = & \alpha_{ij}(1 - M_i^j) \\ m_j^i(\Theta) & = & 1 - \alpha_{ij}(1 - M_i^j) \end{array}$$

Probabilistic based model $^{(3/6)}$

How to find M_i^j?
3th axiom:

3 Conformity to the Bayesian approch (case where $p(S_j|\theta_j)$ is exactly the reality ($\alpha_{ij} = 1$) for all i, j) and all the *a priori* probabilties $p(\theta_i)$ are known)

Probabilistic based model $^{(4/6)}$

Model 1: $M_i^j = \dfrac{R_j p(S_j|\theta_j)}{1 + R_j p(S_j|\theta_j)}$

$$
\begin{aligned}
m_j^i(\theta_i) &= \frac{\alpha_{ij} R_j p(S_j|\theta_j)}{1 + R_j p(S_j|\theta_j)} \\
m_j^i(\theta_i^c) &= \frac{\alpha_{ij}}{1 + R_j p(S_j|\theta_j)} \\
m_j^i(\Theta) &= 1 - \alpha_{ij}
\end{aligned}
$$

with $R_j \geq 0$ a normalization factor.

Probabilistic based model $^{(5/6)}$

Model 2: $M_i^j = R_j p(S_j|\theta_j)$

$$\begin{array}{rcl} m_j^i(\theta_i) & = & 0 \\ m_j^i(\theta_i^c) & = & \alpha_{ij}(1 - R_j p(S_j|\theta_j)) \\ m_j^i(\Theta) & = & 1 - \alpha_{ij}(1 - R_j p(S_j|\theta_j)) \end{array}$$

with $R_j \in [0, (\max_{S_j,i}(p(S_j|\theta_j)))^{-1}]$

In practical:

- α_{ij}: discounting coefficient fixed near 1 and $p(S_j|\theta_j)$ can be given by the confusion matrix
- Adapted to the cases where we learn one class against all the others

Probabilistic based model $^{(6/6)}$

In Malab:

- take the previous confusion matrix or mat_conf=[68 12 22 ; 9 42 5 ; 8 2 87]
- mat_masse= bbaType(mat_conf,alpha,model): gives all the possible bbas (*i.e.* number of classes) for the given confusion matrix, alpha (a constant such as 0.95) and the model (1 or 2)
- bbas=buildBbas(Stst,mat_conf,alpha,model): gives the bbas resulting of the founded classes ginven in Stst

Probabilistic vs Distance

Difficulties:

- Appriou: learning the probabilities $p(S_j|\theta_j)$
- Denœux: choice of the distance $d(x, \mathbf{x}_i)$

Easiness:

- $p(S_j|\theta_j)$ easer to estimate on decisions with the confusion matrix of the classifiers
- $d(x, \mathbf{x}_i)$ easer to choose on the numeric outputs of classifiers (ex.: Euclidian distance)

References

- Toolboxes: http://www.bfasociety.org/wiki/index.php/Toolboxes
- a lot of papers on: http://www.bfasociety.org/wiki/extensions/Wikindx/wikindx3/index.php

References

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