Understanding the large family of Dempster-Shafer theory's fusion operators – a decision-based measure

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Abstract - *Distances between fusion operators are measured using a class of random belief functions. With similarity analysis, the structure of this family is extracted, for two and three information sources. The conjunctive operator, quick and associative but very isolated on a large discernement space, and the arithmetic mean are identified as outliers, while the hybrid method and six proportional conflict-redistributing rules (PCR) form a continuum. The hybrid method is showed as being central for the family of fusion methods. All the fusion operators tested with random belief functions are validated on the fusion of radar data classifiers, and show the interest of some new PCR methods.*

Keywords: Clustering, Dempster-Shafer theory, PCR rules, dissimilarity, random belief functions

1 Introduction

The Dempster-Shafer theory has given birth to a large family of operators, making the fusion between two or more belief functions. Two different operators will very often build two different belief functions from the same entries. They will be reduced later in the treatment chain to a simple decision. Usually one will consider some function defined on the inclusion lattice (credibility, plausibility or pignistic probability) and take its maximum on the discernment space, an anti-chain of the lattice.

Usually, both discernment space and focal elements of the belief functions are just the singletons of Θ. In this situation, we can measure the difference between fusion methods by the differences between the decision they induce, even – and mostly – when there is no prior knowledge of a reality.

We try to use a panel as large as possible of fusion combination methods; feed them with random belief functions, and obtain a clear structure of the known operators.

We first present the seven fusion operators we will compare, and the measure we apply on them. We present briefly what similarity analysis is, and give the structures obtained with the distances built with two or three random belief functions, on two to five classes. We conclude with a short application on radar data.

2 The panel of DST fusion combinations

We place ourselves in the powerset 2^Θ, where Θ is the discernment space. A valid *belief function* allows any $X \subseteq \Theta$ as a focal element. A mapping m on 2^Θ is a belief function if and only if:

$$
\begin{aligned}
&\text{(i)} \quad \forall X \subseteq \Theta, m(X) \in [0,1] \\
&\text{(ii)} \quad \sum_{X \subseteq \Theta} m(X) = 1
\end{aligned}
$$

Sets X such that $m(X) > 0$ are called *focal elements*.

2.1 Usual methods

The *non-normalized conjunctive rule*, given by Smets [10], gives:

$$
m_{\text{Conj}}(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j), \tag{1}
$$

where $Y_j \in 2^\Theta$ is a response of the information source j, and $m_j(Y_j)$ the associated belief function.

The normalized version multiplies all the masses, except the mass on $\emptyset$, by $\frac{1}{1-m(\emptyset)}$. Both versions are associative. Let $\text{Conj}(e_1, ..., e_n)$ be the belief function obtained by the fusion of n belief functions e_i by the conjunctive rule. We have:

$$
\begin{aligned}
\text{Conj}(\text{Conj}(e_1, e_2), e_3) &= \text{Conj}(\text{Conj}(e_1, e_3), e_2) \\
&= \text{Conj}(e_1, e_2, e_3)
\end{aligned}
$$

The *autoconflict* is defined as the conflict generated by a belief function e with the Conj rule: $a(e) = m_{\text{Conj}(e,e)}(\emptyset)$. Fusing n identical belief functions defines the autoconflict of order n, with $n \geqslant 1$:

$$
a_n(e) = m_{\text{Conj}(\underbrace{e, \ldots, e}_{n \text{ times}})}(\emptyset)
$$

This leads to $a_n(e) \leqslant a_{n+1}(e)$. The conjunctive rule is not idempotent: if the belief function e has an auto-conflict of 0, there exists a maximal $Y \subseteq \Theta$ such that for any focal element X of e, $X \subseteq Y$. The conjunctive combination of n times the belief function e tends to $\bar{e}(Y) = 1$ when n tends towards

∞. If $a(e) > 0$ this conjunctive combination tends to $\bar{e}(\emptyset) = 1$, and $\lim_{n \to \infty} a_n(e) = 1$.

The *arithmetic mean* is another simple fusion operator. It is not associative, but belief functions can easily be fused with the result of the Mean operator: $\text{Mean}(e_1, e_2, e_3) = \frac{2}{3}\text{Mean}(e_1, e_2) + \frac{e_3}{3}$. The arithmetic mean is idempotent: $e = \text{Mean}(e, e)$; it works like a weighted vote method:

$$m_{\text{Mean}}(X) = \frac{1}{M}\sum_{j=1}^{M} m_j(X). \qquad (2)$$

The mixed rule was given by Dubois and Prade [3] for the powerset and extended by Dezert and Smarandache to the hybrid rule for the *hyper-powerset* D^Θ (closure of Θ under intersection and union operators in which an equivalence class of $\emptyset$ is defined). It distributes the partial conflict on partial ignorance:

$$m_{\text{DP}}(X) = m_{\text{Conj}}(X) + \sum_{\substack{Y_1 \cup \ldots \cup Y_M = X \\ Y_1 \cap \ldots \cap Y_M = \emptyset}} \prod_{j=1}^{M} m_j(Y_j). \qquad (3)$$

This rule, like all the rules given in the next section, is not associative. We can have $\text{DP}(\text{DP}(e_1, e_2), e_3)$ different of $\text{DP}(e_1, e_2, e_3)$.

2.2 Conflict-redistributing methods

Dezert and Smarandache proposed a list of proportionnal conflict redistribution methods [9] to redistribute the local conflict on the focal elements implied in the local conflict.

The most efficient is the fifth PCR rule given in this paper. Its expression for two belief functions is given in (4) and leads to the generalized rules PCR5 and PCR6, presented thereafter. We use the term PCR for the common restriction of PCR5 or PCR6 on two belief functions.

$$m_{\text{PCR}}(X) = m_{\text{Conj}}(X) + \sum_{\substack{Y \in D^\Theta, \\ X \cap Y \equiv \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right), \qquad (4)$$

where $m_{\text{Conj}}(.)$ is the conjunctive rule given by the equation (1).

Dezert and Smarandache proposed an extension to more than two information sources [9]:

$$m_{\text{PCR5}}(X) = m_{\text{Conj}}(X) + \sum_{i=1}^{M} m_i(X) \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \mathbb{1}_{j>i} \right) \prod_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{\sum_{\substack{Z \in \{X, Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}}} \prod_{Y_{\sigma_i(j)}=Z} \left(m_{\sigma_i(j)}(Y_{\sigma_i(j)}).T(X=Z, m_i(X)) \right)},$$

where σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j + 1 & \text{if } j \geq i, \end{cases} \qquad (5)$$

and:

$$\begin{cases} T(B, x) = x & \text{if } B \text{ is true}, \\ T(B, x) = 1 & \text{if } B \text{ is false}, \end{cases} \qquad (6)$$

This function allows us to make conditional multiplications. We can also write $T(B, x)$ by using the indicator function: $x + \mathbb{1}_B(1 - x)$.

We propose another extension to more than two information sources [6]. PCR5 and PCR6 coincide on the two information sources case.

$$m_{\text{PCR6}}(X) = m_{\text{Conj}}(X) + \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right),$$

where σ is defined like in (5).

This rule can be parametrized to decrease or increase the influence of many small values toward one large one. The first way is given by PCR6f_α, applying the function $f(x) = x^\alpha$ with $\alpha \geq 0$ on each belief value implied in the partial conflict. Any non-decreasing positive function defined on $]0, 1]$ can be used.

$$m_{\text{PCR6f}_\alpha}(X) = m_{\text{Conj}}(X) + \sum_{i=1}^{M} m_i(X)^{1+\alpha} \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X)^\alpha + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})^\alpha} \right)$$

The second way, given by PCR6g_β is to apply the function on the sum of belief functions given to a focal

element. The function used is still $f(x) = x^\beta$, with β positive.

$$m_{\text{PCR6g}_\beta}(X) = m_{\text{Conj}}(X) + \sum_{i=1}^{M} m_i(X) \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)}, \dots, Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}}$$

$$\frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)\left(\prod_{\substack{j>i \\ Y_{\sigma_i(j)}=X}} \mathbb{1}_{j>i}\right)\left(m_i(X) + \sum_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)^\beta}{\sum_{Z \in \{X, Y_{\sigma_i(1)}, \dots, Y_{\sigma_i(M-1)}\}} \left(\sum_{Y_{\sigma_i(j)}=Z} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) + m_i(X)\mathbb{1}_{X=Z}\right)^\beta}$$

3 Decision rule

There are many ways to provide a decision from a belief function. Usually, the maximum of the plausibility, the credibility or the pignistic probability is taken on the space of admissible decisions, an anti-chain Γ of the lattice 2^Θ or D^Θ. Γ is an anti-chain if for any X and any Y in Γ, we cannot have $X \subsetneq Y$ nor $Y \subsetneq X$.

Here, we only use these functions in 2^Θ. Their extension to D^Θ is immediate for plausibility and credibility. For the pignisitc probability, one may refer to [2].

$$\text{bel}(X) = \sum_{Y \subseteq X, Y \neq \emptyset} m(Y) \qquad (7)$$

$$\text{pl}(X) = \sum_{Y \subseteq \Theta, Y \cap X \neq \emptyset} m(Y) \qquad (8)$$

$$\text{betP}(X) = \sum_{Y \subseteq \Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)} \qquad (9)$$

For any $X \subseteq \Theta$, we have: $\text{bel}(X) \leqslant \text{betP}(X) \leqslant \text{pl}(X)$, even with $m(\emptyset) = 0$ (closed world hypothesis). Here, we consider the maximum of pignistic probability.

Notice that the input belief functions we use only have singletons and Θ as focal elements, so the obtained belief functions, except with the Dubois and Prade method, only have singletons, Θ and $\emptyset$ as focal elements. For any X and Y subsets of Θ, we have $\text{bel}(X) \leqslant \text{bel}(Y)$ if and only if $\text{betP}(X) \leqslant \text{betP}(Y)$ if and only if $\text{pl}(X) \leqslant \text{pl}(Y)$.

Difference between normalized conjunctive rule and non-normalized conjunctive rule is a multipliative factor of $\frac{1}{1-m(\emptyset)}$. So their pignistic probabilities are equal, and more generally the decision based on the order bel or pl induce on Θ is the same.

4 Random belief functions

In order to compare the different combination rules, we feed them with random belief functions, and compare the decisions taken by the rules. The focal elements of the random belief functions are all the elements of Θ, and Θ itself. We have $m(\Theta) + \sum_{x \in \Theta} m(\{x\}) = 1$.

A random belief function for a discernment space of cardinal n is defined by an uniform probability distribution on $[0,1]^n \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_1^n x_i \leqslant 1\}$.

Subsets of Θ with cardinal higher than 1 are never focal elements, and we have a probability of 0 to get a null mass on a singleton or on indifference, and also a probability of 0 that two focal elements have the same mass, plausibility, credibility or pignistic probability. So we cannot have a total conflict between two random belief functions, which would lead to an error when calculating the pignistic probability of their combination, nor a decision rule concerned by multiple maxima.

5 Modification of decision measure

As truth is not assumed to be available to the performance evaluation, we do not count the good or the bad decisions, but only the similarity between decisions induced by the different fusion operators.

The decision induced by a belief function e is $x_i = \text{dec}(e)$, with $x_i \in \Theta$, such that $\text{betP}(x_i) = \max_{x \in \Theta} \text{betP}(x)$. The dissimilarity $d(R, S)$ between two fusion operators R and S is given by the probability of having $\text{dec}(R(e_1, \dots, e_k)) \neq \text{dec}(S(e_1, \dots, e_k))$, with $e_1, \dots, e_k$ random belief functions on 2^Θ, with the same dicernment space Θ.

Numerical results presented in the following sections are the estimated percentages of these events.

We do not present here approximations of the obtained dissimilarity, but we focus on the structure they induce on the fusion operators [1][4].

A graph $G = (X, E)$ compatible with a dissimilarity d on X has the property that for any vertices x and y, $d(x, y)$ is greater than the largest $d(u, v)$ for u and v in a path from x to y, for at least one path between x and y in G. As a natural cluster [5] of a dissimilarity d is a maximal clique of a threshold graph of d ($G_\lambda = (X, E_\lambda)$ with $E_\lambda = \{(u, v) \in X^2 \mid d(u, v) \leqslant \lambda\}$), the graph G restricted to any natural cluster of d is connected. We use graphs minimal in terms of number of edges for this property in order to obtain a structure as simple as possible: G_d, a minimum rigidity graph of d.

The following figure provides an example of how to build a minimum rigidity graph from a dissimilarity d:

d	x	y	z	t	u
x	0	1	2	3	3
y	1	0	3	2	3
z	2	3	0	1	2
t	3	2	1	0	2
u	3	3	2	2	0

The natural clusters of d are xy and zt (diameter 1), xz, yt and zut (diameter 2) and $xyztu$ (diameter 3). The dissimilarity d admits two minimum rigidity graphs:
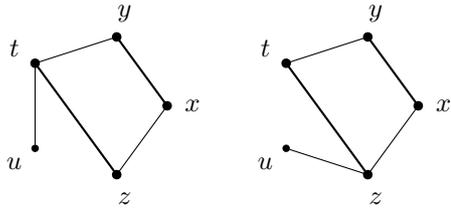
Figure 1. *Rigidity graphs of the natural clusters of d*

Obtaining G_d is NP-hard [1], but we are dealing with only 9 elements – our fusion operators – so this operation is not too difficult. Also, for strongly structured dissimilarities, as all the ones presented in the following sections, having rigidity graphs of $|X|-1$ or $|X|$ edges, polynomial algorithms exist.

A more usual study, through hierarchical classification, would have shown the homogeneity of the proportionnal conflict redistribution rules family, but would not have shown its internal structure.

5.1 Two belief functions

When only two belief functions e_1 and e_2 are fused, we have the following equalities:

$$\mathrm{PCR5}(e_1, e_2) = \mathrm{PCR6}(e_1, e_2) \qquad (10)$$
$$\forall \lambda \ \mathrm{PCR6f}_\lambda(e_1, e_2) = \mathrm{PCR6g}_\lambda(e_1, e_2) \quad (11)$$

With only two classes, we have also:

$$\mathrm{dec}(\mathrm{Conj}(e_1, e_2)) = \mathrm{dec}(\mathrm{DP}(e_1, e_2)) \qquad (12)$$

With two to five classes we obtain the following percentages of decision change, seen as a dissimilarity.

Two classes : $d_{2,2}$					
	Conj	PCR	Mean	PCR6$_{0.5}$	PCR6$_2$
Conj	0.0	0.7	2.2	0.3	1.2
PCR	0.7	0.0	2.9	0.3	0.5
Mean	2.2	2.9	0.0	2.5	3.4
PCR6$_{0.5}$	0.3	0.3	2.5	0.0	0.9
PCR6$_2$	1.2	0.5	3.4	0.9	0.0

Three classes : $d_{2,3}$					
Conj	DP	PCR	Mean	PCR6$_{0.5}$	PCR6$_2$
0.0	3.5	5.5	5.8	4.6	6.9
3.5	0.0	2.3	3.3	1.3	3.7
5.5	2.3	0.0	4.5	1.0	1.4
5.8	3.3	4.5	0.0	3.8	5.5
4.6	1.3	1.0	3.8	0.0	2.4
6.9	3.7	1.4	5.5	2.4	0.0

Four classes : $d_{2,4}$					
Conj	DP	PCR	Mean	PCR6$_{0.5}$	PCR6$_2$
0.0	6.2	9.2	8.7	7.9	11.2
6.2	0.0	3.5	3.8	2.0	5.5
9.2	3.5	0.0	5.5	1.5	2.0
8.7	3.8	5.5	0.0	4.5	7.0
7.9	2.0	1.5	4.5	0.0	3.5
11.2	5.5	2.0	7.0	3.5	0.0

Five classes : $d_{2,5}$					
Conj	DP	PCR	Mean	PCR6$_{0.5}$	PCR6$_2$
0.0	8.4	12.1	11.1	10.5	14.2
8.4	0.0	4.3	3.9	2.5	6.6
12.1	4.3	0.0	5.9	1.8	2.4
11.1	3.9	5.9	0.0	4.8	7.8
10.5	2.5	1.8	4.8	0.0	4.1
14.2	6.6	2.4	7.8	4.1	0.0

With two classes, the order (PCR6$_2$, PCR5&6, PCR6$_{0.5}$, Conj&DP, Mean) is compatible with the distance obtained $d_{2,2}$, which is a robinsonian dissimilarity:

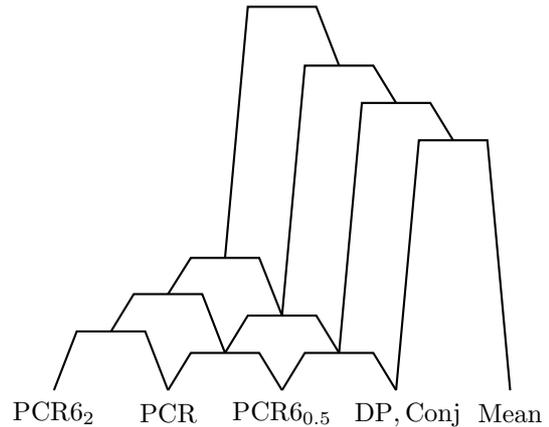PCR6$_2$ PCR   PCR6$_{0.5}$ DP, Conj Mean

Figure 2. *Pyramid representation of $d_{2,2}$*

With more than two classes, decision can differ between DP and Conj: DP fusion method appears between Conj and PCR6$_{0.5}$. Also, the mean operator is more similar to DP than Conj. The following figure represents a tree, compatible with $d_{2,3}$, $d_{2,4}$ and $d_{2,5}$. Edge lengths are an affine transformation of $d_{2,4}$.
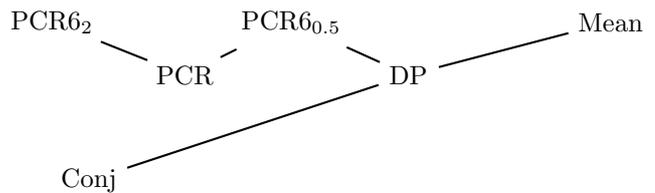

Figure 3. *Rigidity tree of $d_{2,3}$, $d_{2,4}$ and $d_{2,5}$.*

Limited to a discernment space of two classes, the conjunctive rule is very similar to the conflict redistribution rules, and the arithmetic mean is significantly different. With three classes, the conflict-redistributing rules and the Dubois & Prade rule form a natural cluster of diameter 3.7; within this family, the Dubois & Prade rule is the most similar to the outliers, Conj and the arithmetic mean, and those outliers are at similar distances. With more than three classes, the conjunctive rule provides decisions very different from all the other rules.

5.2 Three belief functions

Adding a third belief function creates a difference between PCR5 and PCR6. It also separates the operators PCR6f$_\lambda$ (shortened to P6f$_\lambda$ in the tables) and PCR6g$_\lambda$, shortened to P6g$_\lambda$.

Two classes : $d_{3,2}$							
Conj	$PCR6$	$PCR5$	$Mean$	$P6f_{\frac{1}{2}}$	$P6g_{\frac{1}{2}}$	$P6f_2$	$P6g_2$
0.0	1.0	2.6	3.8	0.8	0.6	2.2	1.6
1.0	0.0	1.9	4.5	1.0	0.4	1.6	0.6
2.6	1.9	0.0	6.3	2.8	2.2	0.8	1.5
3.8	4.5	6.3	0.0	3.5	4.2	6.0	5.0
0.8	1.0	2.8	3.5	0.0	0.7	2.6	1.5
0.6	0.4	2.2	4.2	0.7	0.0	1.9	1.0
2.2	1.6	0.8	6.0	2.6	1.9	0.0	1.2
1.6	0.6	1.5	5.0	1.5	1.0	1.2	0.0

The distance $d_{3,2}$ is compatible with the following graph, which is not a tree: there are two concurrent ways to join the arithmetic mean operator to the PCR family. The vertex Conj is merged with the Dubois & Prade rule.
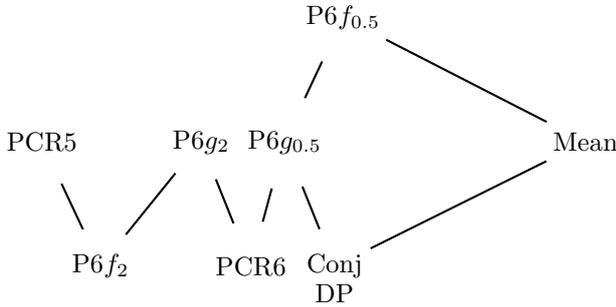

Figure 4. *Rigidity graph of $d_{3,2}$*

Three classes : $d_{3,3}$								
Conj	DP	PCR6	PCR5	Mean	$P6f_{\frac{1}{2}}$	$P6g_{\frac{1}{2}}$	$P6f_2$	$P6g_2$
0.0	4.5	8.5	10.0	8.3	7.0	7.3	10.8	9.9
4.5	0.0	5.2	7.5	4.8	3.4	3.8	7.9	6.8
8.5	5.2	0.0	3.2	6.8	2.2	1.4	3.1	1.8
10.0	7.5	3.2	0.0	9.8	5.2	4.2	1.7	2.3
8.3	4.8	6.8	9.8	0.0	4.9	5.8	9.7	8.3
7.0	3.4	2.2	5.2	4.9	0.0	1.0	5.3	3.9
7.3	3.8	1.4	4.2	5.8	1.0	0.0	4.4	3.2
10.8	7.9	3.1	1.7	9.7	5.3	4.4	0.0	1.5
9.9	6.8	1.8	2.3	8.3	3.9	3.2	1.5	0.0

Four classes : $d_{3,4}$								
Conj	DP	PCR6	PCR5	Mean	$P6f_{\frac{1}{2}}$	$P6g_{\frac{1}{2}}$	$P6f_2$	$P6g_2$
0.0	8.8	13.8	15.7	12.0	11.6	12.0	16.9	16.0
8.8	0.0	7.4	10.5	4.4	4.7	5.4	11.2	10.0
13.8	7.4	0.0	3.8	8.3	3.1	2.1	4.1	2.8
15.7	10.5	3.8	0.0	11.9	6.7	5.6	2.1	2.4
12.0	4.4	8.3	11.9	0.0	5.6	6.6	12.1	10.8
11.6	4.7	3.1	6.7	5.6	0.0	1.1	7.1	5.8
12.0	5.4	2.1	5.6	6.6	1.1	0.0	6.2	4.9
16.9	11.2	4.1	2.1	12.1	7.1	6.2	0.0	1.5
16.0	10.0	2.8	2.4	10.8	5.8	4.9	1.5	0.0

The distances $d_{3,3}$ and $d_{3,4}$ are almost compatible with the following tree, where edge lengths are an affine trasform of $d_{3,3}$. To be complete, we have to add an edge to make a path between Conj and PCR5 not passing through PCR6f$_2$; there are five possibilities to add such an edge, which has a large weight and does not provide many structural information.
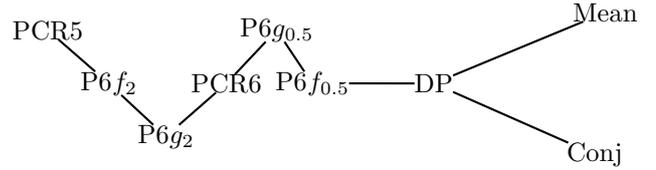

Figure 5. *Stable edges of the rigidity graphs of $d_{3,3}$ and $d_{3,4}$.*

5.3 Using a non-associative rule by pairs

Mean and conjunctive rule are associative: making the fusion of M belief functions through those rules is not more costly than M times the fusion of two belief functions.

But the fusion of M belief functions, each one having p focal elements, needs $\mathcal{O}(n^M)$ operations for the other operators. For each fusion operator R, We measure the probability of having $\text{dec}(R(R(e_i, e_j), e_k)) \neq \text{dec}(R(e_1, e_2, e_3))$ with (i, j, k) any permutation of $(1, 2, 3)$.

Classes	2	3	4
PCR6	3.1	6.8	10.6
PCR5	4.6	8.4	12.0
PCR6f$_2$	4.7	7.0	9.0
PCR6f$_{\frac{1}{2}}$	5.1	11.5	17.7
DP	15.9	23.0	26.5

The Dubois and Prade rule is the most sensitive to the order of parameters when calculating $\text{dec}(\text{DP}(\text{DP}(e_i, e_j), e_k))$. Note that when the only focal elements are the singletons and indifference:

$$\text{dec}(\text{DP}(e_1, \ldots, e_M)) = \lim_{\varepsilon \to 0} \text{dec}(PCR6g_\varepsilon(e_1, \ldots, e_M)).$$

For the other rules, using pairwise operators instead of an operator on three belief functions leads to differences in term of decision greater than using an associative operator like Conj or a low time-consumming method like Mean.

Non-associative operators cannot be safely considered as associative to speed up a fusion system.

6 Classes of fusion operators

We consider the classes formed by the distances $d_{3,3}$ and $d_{3,4}$, as the other distances either do not distinguish between PCR5 and PCR6 or between DP and Conj.

The Dubois and Prade fusion method, replacing local conflict on local indifference, is central. It makes a connection between the conflict-redistributing rules (PCRs) and the conjunctive rule, placing local conflict on $\emptyset$ and the arithmetic mean, which has a simple additive principle, and does not generate conflict.

6.1 Two outliers: conjunctive rule and mean

The conjunctive rule is multiplicative, so one very low weight on a singleton is sufficient to reduce the chances of this singleton of being decided nearly to zero. This is the key of the Zadeh paradox [11], and explains the large difference between the decision proposed by the conjunctive rule and the other rules.

As we use the maximum of pignistic probability, many rules similar to Conj are equivalent: by example the normalized conjunctive rule or putting local conflict on Θ instead of $\emptyset$.

The arithmetic mean does not work on the notion of conflict. Nevertheless, its nearest neighbour is constant for all the situations studied: it is the Dubois and Prade method, and the distance between this outlier and the PCR family is more stable than the conjunctive rule's one.

6.2 A chain to sort PCR methods

Among the PCR rules, an order shows up: (PCR5, PCR6f$_2$, PCR6g$_2$, PCR6, PCR6g$_{0.5}$, PCR6f$_{0.5}$).

We have seen that with the focal elements used in our belief functions, Dubois and Prade rule is the limit in 0 of the $PCR6g_\varepsilon$, explaining the right part of this order.

The value of 2.5 for λ parameter minimizes $\mathrm{dist}(PCR6f_\lambda, PCR5)$: PCR5 is not a limit for this generalization of PCR6. Using more values for λ brings the internal structure of PCR methods:
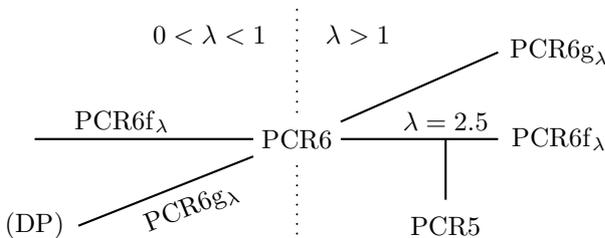

Figure 6. *Variations around PCR6*

7 Application to radar data

We treat radar frequency profile data, composed of 1500 elements, with three classifiers and ten targets: Supervised ART, k-fuzzy nearest neighbours, and multi-layer perceptron [7]. We convert the inner masses of these classifiers into belief functions, with a normalization that guarantees that the mean indifference of each belief function is 0.2, and that there is no more than three focal classes, indifference included. To obtain significant measure of good classification rate, we use 800 different pairs of learning set (size 1000) and testing set (size 500). By the way we have have 400000 distinct belief functions for each classifier to feed the fusion methods.

The best results are obtained for PCR6, PCR6g$_{0.5}$, PCR6f$_{0.5}$, DP and Mean, which form a connected class of $G_{d_{3,3}}$ and $G_{d_{3,4}}$. Maximum is reached for PCR6f$_{0.5}$. The following table gives the percentages of good classification obtained with the different fusion methods. A confidence level at 95% gives a interval of $\pm 0.1\%$ around each value.

Conj	DP	Mean	PCR6	PCR5	P6f$_{\frac{1}{2}}$	P6g$_{\frac{1}{2}}$	P6f$_2$	P6g$_2$
89.83	89.99	90.09	90.05	89.85	90.11	90.08	89.94	90.00

Dissimilarity obtained on radar data								
Conj	DP	Mean	PCR6	PCR5	P6f$_{\frac{1}{2}}$	P6g$_{\frac{1}{2}}$	P6f$_2$	P6g$_2$
0.0	0.68	1.34	2.02	2.83	1.53	1.60	2.77	2.53
0.68	0.0	0.85	1.47	2.37	0.94	1.04	2.27	2.01
1.34	0.85	0.0	1.59	2.65	1.01	1.22	2.44	2.09
2.02	1.47	1.59	0.0	1.08	0.61	0.44	0.88	0.55
2.83	2.37	2.65	1.08	0.0	1.67	1.46	0.51	0.71
1.53	0.94	1.01	0.61	1.67	0.0	0.23	1.49	1.15
1.60	1.04	1.22	0.44	1.46	0.23	0.0	1.29	0.99
2.77	2.27	2.44	0.88	0.51	1.49	1.29	0.0	0.39
2.53	2.01	2.09	0.55	0.71	1.15	0.99	0.39	0.0

The proportions of decision change between the different fusion methods are compatible with the structures showed by the random belief functions. The rigidity graphs compatible with this dissimilarity are the same than the rigidity graphs of $d_{3,3}$ and $d_{3,4}$, shown figure 5.

8 Conclusion

This cartography of these usual or new fusion methods shows a strong (as it is in most cases a tree) structure. This map should be used as a guide to test methods in the neighboring of a efficient one to improve the fusion results. It can also be taken by leaves and root, testing Mean, Conj, PCR5 and DP to identify a promising branch, and explore it in order to find an efficient fusion method for a given dataset.

The third way may be the more important for a young concept like PCR rules: it "shows" unexplored fusion methods, lying between DP and Mean, or someway orthogonal to the chain from PCR5 to DP. In order to improve results on radar data, we may have to build fusion methods between PCR6 and Mean, significantly different of DP.

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