

Comparative study of high order statistics estimators

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Abstract: To characterize and process a signal, many high order statistics are used by the signal processing researchers. Specific features of the data (temporal, stationary) and real time applications require the development of new estimators. In this paper, we study some estimators of high order moment and cumulant using adapted to different kind of signals.

Key words: Estimators, high order statistic, moment, cumulant, temporal and non-stationary data.

1. INTRODUCTION

Some specific features of data and problem in signal processing field require the study of different estimator classes. Indeed, data could be temporary, stationary or not. Beside that in various applications, the processing should be in real time. In example, for acoustic signals (sonar or speech [1,2]), data are considered as non-stationary for long period and can be considered stationary within few milliseconds. However, background noise added by the used sensor is often considered as stationary. In speech/noise detection or recognition applications, the processing should be done in real or at least pseudo real time. Therefore, quick and efficient estimations of the signal statistics among other parameters are deeply needed in various techniques such as classification, detection, recognition, and sources separation, etc.

In many signal processing applications, researchers as well engineers assume that signal distributions are Gaussian or Laplacian in order to simplify the calculus [2]. Indeed, this assumption means that a signal distribution can just be given by only its mean and its standard deviation. However, this strong assumption can not be satisfied in various recent applications. Since the last two decades, other statistical information have been introduced as asymmetric and flatness estimators (given for example by the skewness and kurtosis) or more generally High Order Statistics [3,4,5,6,7]. Thus many estimators have been proposed [2,8,9,10].

After brief introduction on moments and cumulants in section 2, we study different estimators of moments and cumulants in section 3. Section 4 presents a comparative study of these estimators. Section 5 presents our conclusions on estimators of high order statistics for signal processing.

2. THEORETICAL BACKGROUND

Let X denotes a stochastic process in a real space, describing a signal, its characteristic function is given by:

$$\phi_X(t) = \int_{-\infty}^{+\infty} \exp(itx) p_X(x) dx \quad (1)$$

This function is continuous complex function. We should mention that its module is less or equal to 1, and that $\Phi_X(0) = 1$. Using the previous equation, one can define the second characteristic function as:

$$\varphi_X(t) = \ln(\phi_X(t)) \quad (2)$$

By definition, the q th order moment is given [6,7] by the q th order derivative of the first characteristic function around zero:

$$\mu_q = (-1)^q \cdot \left. \frac{d^q \phi_X(t)}{dt^q} \right|_{t=0} = E[X^q] \quad (3)$$

By similar definition, the q th order cumulant is given as the q th order derivative of the second characteristics function at the origin:

$$\kappa_q = (-1)^q \cdot \left. \frac{d^q \varphi_X(t)}{dt^q} \right|_{t=0} = \text{Cum}[X, X, \dots, X] \quad (4)$$

It is clear that the first and second order cumulants are respectively the mean and the variance of X . In the case of Gaussian distribution, we should mention that all cumulants with order higher than 2 are null.

Leonov and Shirayev wrote down general relationships among moments and cumulants. According to their study, a q th order cumulant can be evaluated as:

$$\text{Cum}_q(X) = \text{Cum}[X, X, \dots, X] = \sum_{p=1}^q (-1)^p (p-1)! \mu_{v_1} \mu_{v_2} \dots \mu_{v_p} \quad (5)$$

where the numbers $\{v_1, v_2, \dots, v_p : 1 \leq p \leq q\}$ are such as $v_i \in \{1, \dots, q-p+1\}$ and $\sum_{i=1}^p v_i = q$. We should mention here that in their original study, they developed the relationships in the case of q random variables [7]. Equation (5) can be easily obtained from the original relationship. Using equation (5), one can write the fourth order cumulant as in [8]:

$$\text{Cum}_4(X) = \mu_4 - 4\mu_1\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4, \quad (6)$$

Last equation can be simplified for a zero mean signal:

$$\text{Cum}_4(X) = \mu_4 - 3\mu_2^2. \quad (7)$$

3. HIGH ORDER STATISTICS ESTIMATORS

3.1 Arithmetic estimators

Let us consider N realizations x_i of a stochastic process X assumed to be an ergodic one. In this case, the arithmetic estimator of the q th order moment is given by:

$$\hat{\mu}_q = \frac{1}{N} \sum_{i=1}^N x_i^q. \quad (8)$$

This estimator means that the signal X is stationary over N samples. This estimator is a non biased estimator and its variance is given by:

$$\text{var}(\hat{\mu}_q) = \frac{1}{N} (\mu_{2q} - \mu_q^2), \quad (9)$$

we can notice that this estimator is a consistent one; hence more the temporal signal will be stationary more the estimation will be better.

An arithmetic estimator of the q th order cumulant can be developed from equation (5):

$$\widehat{\text{Cum}}_q(X) = \sum_{p=1}^q (-1)^p (p-1)! \hat{\mu}_{v_1} \hat{\mu}_{v_2} \dots \hat{\mu}_{v_p}. \quad (10)$$

Unfortunately, this estimator is biased, since for example the estimator $\hat{\mu}_{v_1} \hat{\mu}_{v_2}$ of $\mu_{v_1} \mu_{v_2}$ is obviously biased. In a general case, we have:

$$E[\widehat{\text{Cum}}_q(X)] = \sum_{p=1}^q \frac{(-1)^p}{N^{p-1}} (p-1)! \begin{pmatrix} \mu_q \\ +(N-1)\mu_{v_{k1}}\mu_{v_{k2}} \\ +(N-1)^2\mu_{v_{k1}}\mu_{v_{k2}}\mu_{v_{k3}} \\ +\dots+(N-1)^{p-1}\mu_{v_1}\dots\mu_{v_p} \end{pmatrix}, \quad (11)$$

with $\sum_{i=1}^p v_i = q$ at each row, for example in the second term $\mu_{v_{k1}} \mu_{v_{k2}}$, v_{k1} and v_{k2} represent the sum of two subsets of the indices set $\{v_1, v_2, \dots, v_p\}$. Otherwise, This estimator is consistent. A non-biased cumulant estimator can be deduced from (11):

$$\widehat{\text{Cum}}_q(X) = \sum_{p=1}^q c_p (-1)^p (p-1)! \hat{\mu}_{v_1} \hat{\mu}_{v_2} \dots \hat{\mu}_{v_p}, \quad (12)$$

where c_p are constants depending on the partitions of the indices v_i . To evaluate these constants, one should solve an equation system with a number of equations equal to the number of unknowns. We can demonstrate that such a system has a unique solution. To fix our ideas, let us consider the fourth order cumulants case:

$$\widehat{\text{Cum}}_4(X) = a\hat{\mu}_4 - 4b\hat{\mu}_1\hat{\mu}_3 - 3c\hat{\mu}_2^2 + 12d\hat{\mu}_1^2\hat{\mu}_2 - 6e\hat{\mu}_1^4, \quad (13)$$

with

$$\begin{cases} a = \frac{N^3 + N^2 - 24N + 24}{(N-1)(N-2)(N-3)}, b = \frac{N(2N^2 - 10N + 9)}{2(N-1)(N-2)(N-3)}, \\ c = \frac{N(N^2 - N - 6)}{(N-1)(N-2)(N-3)}, d = \frac{N^2(2N-5)}{2(N-1)(N-2)(N-3)}, \\ e = \frac{N^3}{(N-1)(N-2)(N-3)}. \end{cases} \quad (14)$$

In the case of zero mean signals, we obtain easily [8]:

$$\widehat{\text{Cum}}_4(X) = \frac{N+2}{N-1} \hat{\mu}_4 - 3 \frac{N}{N-1} \hat{\mu}_2^2. \quad (15)$$

The arithmetic estimator of the q th order moments or cumulants such we write them here are not adaptive. This point is an important one for real time applications. Estimator (8) can be modified to an adaptive form [8]:

$$\hat{\mu}_q(k) = \frac{1}{k} \sum_{i=1}^k x_i^q = \frac{(k-1)\hat{\mu}_q(k-1) + x_k^q}{k}, \quad (16)$$

with $k > 1$ and $\hat{\mu}_q(1) = x_1^q$. Estimator (16) converges very fast, however it has been shown [8] that it is a good estimator for non-stationary signals. The cumulant estimator can be calculated from (12), this suppose to put in a memory the q values $\hat{\mu}_i(k-1)$ with $i \in \{1, \dots, q\}$.

3.2 Exponential estimators

The exponential estimators are defined as following:

$$\hat{\mu}_q = (1 - \lambda_q) \sum_{i=1}^N \lambda_q^{N-i} x_i^q, \quad (17)$$

where $0 < \lambda_q < 1$ stands for a forgotten factor which its value depends on the order of the estimated statistic. This estimator can be calculated easily in an adaptive way ($\hat{\mu}_q(k) = \lambda_q \hat{\mu}_q(k-1) + (1 - \lambda_q) x_k^q$), but it is biased ($E[\hat{\mu}_q] = (1 - \lambda_q^N) \mu_q$), and it is asymptotically non biased one. The main interest of such estimator reside on the fact that it can give better estimation for the moments of non-stationary signals. Thus more λ_q is close to 1, more the past samples are taking into account. A non biased exponential estimator can be:

$$\hat{\mu}_q = \frac{(1 - \lambda_q)}{(1 - \lambda_q^N)} \sum_{i=1}^N \lambda_q^{N-i} x_i^q. \quad (18)$$

Then an adaptive exponential estimator is given by:

$$\hat{\mu}_q(k) = \frac{1}{(1 - \lambda_q^k)} \left(\lambda_q (1 - \lambda_q^{k-1}) \hat{\mu}_q(k-1) + (1 - \lambda_q) x_k^q \right). \quad (19)$$

The cumulants are estimated using (12) and (19). The authors of [9] proposed an adaptive fourth order moment estimator for zero mean signals:

$$\widehat{Cum}_4(X)(k) = \widehat{Cum}_4(X)(k-1) + (1 - \gamma) H_k(\widehat{Cum}_4(X)(k-1)) \quad (20)$$

with γ a forgotten factor and

$$H_k(x) = x_k^4 - 3x_k^2 \hat{\mu}_2(k-1) - x \quad (21)$$

Let us consider the comparison between this estimator and the following one:

$$\widehat{Cum}_4(X)(k) = x_k^4 - 3\hat{\mu}_{k-1}^2. \quad (21)$$

The adaptive estimator (20) is asymptotically non biased, but it has a slow convergence [8]. We can write this estimator in general case by:

$$H_k(x) = x_k^4 - 4x_k^3 \hat{\mu}_1(k-1) - 3x_k^2 \hat{\mu}_2(k-1) + 12x_k^2 \hat{\mu}_1^2(k-1) - 6\hat{\mu}_1^4(k-1) - x \quad (22)$$

3.3 Mixed estimator

It is clear that exponential estimator (19) can deal with non-stationary signals, however the influence of past samples can become much important than the influence of the more recent

samples. In order to solve such problem for weakly stationary signal (*i.e.* over $n < N$ samples, as in sonar applications [1]), we propose the following new estimator:

$$\hat{\mu}_q = \frac{1}{n+1} \sum_{i=N-n}^N x_i^q + (1 - \lambda_q) \sum_{i=1}^{N-n-1} \lambda_q^{N-n-1-i} x_i^q = \hat{a}_q + \hat{e}_q \quad (23)$$

This mixed estimator is the sum of an arithmetic estimator \hat{a}_q over the previous n samples and an exponential estimator \hat{e}_q on the other samples. The mixed estimator can become non biased, if one consider the following relation instead of (24):

$$\hat{\mu}_q = \frac{\hat{a}_q + \hat{e}_q}{(2 - \lambda_q^{N-n-1})} \quad (24)$$

An adaptive version can be given by:

$$\left\{ \begin{array}{l} \hat{e}_q(k) = \lambda_q \hat{e}_q(k-1) + (1 - \lambda_q) x_k^q \\ \hat{\mu}_q(k) = \frac{(2 - \lambda_q^{k-n-2})}{2 - \lambda_q^{k-n-1}} \left(\hat{\mu}_q(k-1) + \frac{1}{n} x_k^q + (\lambda_q - 1) \hat{e}_q(k-1) \right) \\ \quad + (1 - \lambda_q) x_k^q - \frac{1}{n(2 - \lambda_q^{k-n-2})} x_{k-n-1}^q \end{array} \right. \quad (25)$$

3. COMPARATIVE STUDY

In order to compare the different estimators (16), (19), and (25) of the fourth order moment, some experimental results are presented hereinafter. Unfortunately, we couldn't evaluated the new proposed mixed estimator on real weakly stationary data such as sonar data since the theoretical values are unknown. However, the comparison among fourth order cumulants estimators was done using estimated moments by arithmetic estimator ((15) (made for zero mean signals), (13) (made for general case)), by exponential estimator (20) and the simplified one (21), and finally by estimator (22). The performances of these estimators are evaluated on three kind of simulated signals: a zero mean stationary signal (Figure 1), a non-centered stationary signal (Figure 2) and a non-stationary centered signal (Figure 3).

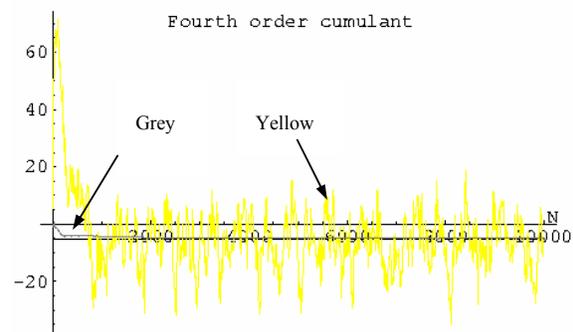
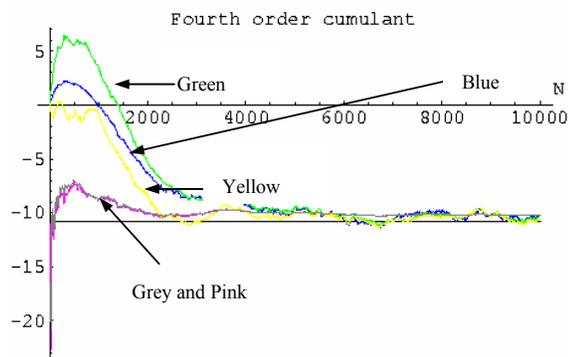
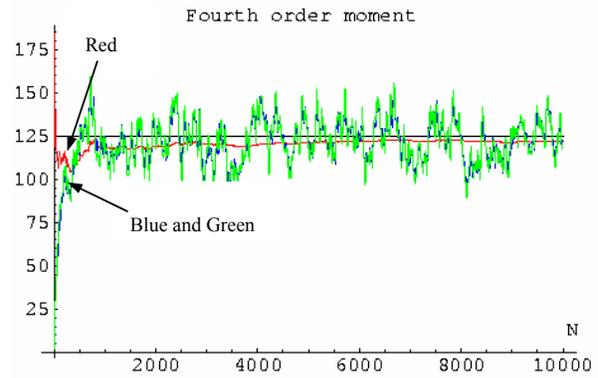
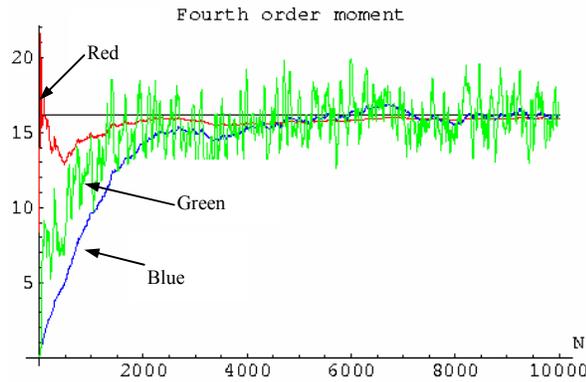


Figure 1. Fourth order moment and cumulant estimators of a centered and stationary signal (type white noise), the forgotten factor is 0.999 for all moments and $\gamma = 0.997^1$.

Figure 2. Fourth order moment and cumulant estimators of a non-centered and stationary signal (type white noise), the forgotten factor was 0.99 for all moments and $\gamma = 0.997^1$.

¹ The previous the figures represent the estimation of the moments with respect to the samples number as following:

- In black the theoretical value
- In red, arithmetic estimator (16)
- In blue, exponential estimator (19)
- In green, mixed estimator (25) with $n=50$

Concerning the estimation of the cumulants, different colors have been used to represent different estimators:

- In black, theoretical values
- In pink, estimator (15).
- In grey, estimator (13) for a centered or non centered signal.
- In blue, estimator (21) for a centered signal.
- In green, estimator (20) for a centered signal.
- Finally, in yellow, estimator (22) for a centered or non centered signal.

Figures 1 and 2 show the fact that the arithmetic estimator converges faster than the exponential and the mixed estimators on stationary signals (with or without zero mean samples). One can also notice that the variance of the mixed estimator is more sensitive to the value of the forgotten factor than the exponential estimator. For a well chosen value of the forgotten factor, the performances of exponential and mixed estimator are very similar. Figure 3 shows that the arithmetic estimator can not estimate the moments of the non-stationary signal, unlike the exponential and mixed estimators.

The comparison among the different estimators of the fourth order cumulant shows that the arithmetic estimator converges more quickly with a small variance on stationary signals. On the other hand, exponential estimators are characterized by a slow convergence and a high variance which means that such estimators should be avoided in the case of stationary signals. Moreover we remark that the generalization (22) of the estimator (20) for a stationary and centered signal has a good and fast convergence speed (Figure 1). Estimator (21) gives also good results on stationary and centered signals.

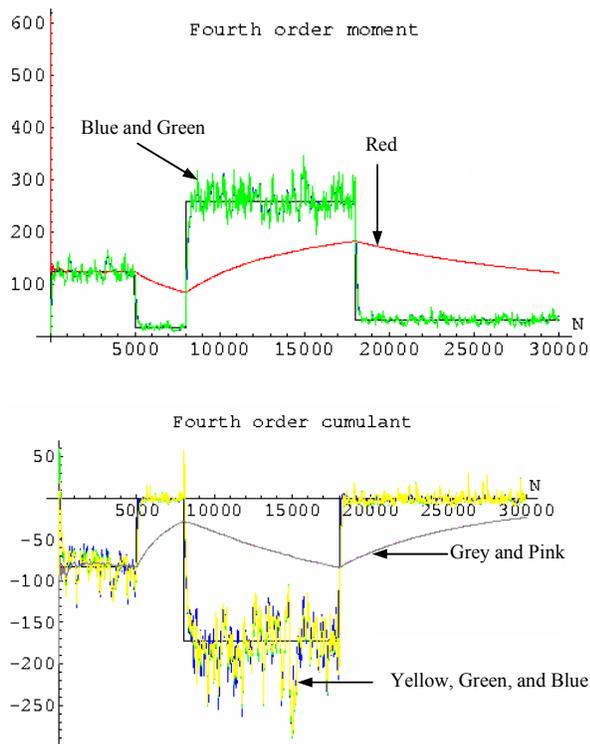


Figure 3. Fourth order moment and cumulant estimators of a centered and non-stationary signal (type white noise), the forgotten factor was 0.99 for all moments and $\gamma = 0.99^1$.

Figure 3 shows the fact that the arithmetic estimator of the cumulants is not a good estimator for non-stationary signals. The performances of the three other estimators are very similar, however figure 1 shows that estimator (22) converges more quickly than the other.

5. CONCLUSION

In this paper, we have compared three estimators of moments and five estimators of cumulants. The estimators based on the arithmetic form are well adapted for stationary signals. However, stationarity properties can not be satisfied in many applications. Thus the estimators based on the exponential form must be preferred for non-stationary signals; on the other hand they converge less quickly, they have high variances and they are sensitive to the chosen value of the forgotten factor. More the order is high, more their convergences become a problem.

Thus the choice of the estimator of high order statistics must be done according to the kind of signal and the application. Finally, we can conclude that a unique estimator with good convergence and small variance for all kind of signals actually does not exist.

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